Ex 2C

1 Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

By example 2.36 $\dim(\mathbb{R}^2)$. Let U be a subspace of $\mathbb{R}^2$

2.37 *dimension of a subspace*

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Then, $\dim(U) \leqslant 2$. Thus $\dim(U) = 0, 1, 2$.

- Case-I) If $\dim(U) = 0$.

We know that $\dim(\{0\}) = 0$. Here $\{0\}$ is the trivial subspace.

2.39 *subspace of full dimension equals the whole space*

Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

Then, $U = \{0\}$.

Case-II) If $\dim(U) = 1$,

Then basis of U have only one non-zero vector. let say that vector $0 \neq x \in U$. Then,

$$U := \{kx \mid k \in \mathbb{R}\}$$

Note that U is the line go through x and the origin.

- Case-III : If $\dim(U) = 2$.

We know that $\dim(\mathbb{R}^2) = 2$.

2.39 *subspace of full dimension equals the whole space*

Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

Then $U = \mathbb{R}^2$.

Therefore, subspace of $\mathbb{R}^2$ are
- $\{0\}$
- all the line go through the origin.
- $\mathbb{R}^2$

$\blacksquare$.

2 Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all lines in $\mathbf{R}^3$ containing the origin, all planes in $\mathbf{R}^3$ containing the origin, and $\mathbf{R}^3$.

By examp 2.36., $\dim(\mathbb{R}^3) = 3$.

2.36 example: *dimensions*

- $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.

Let U be subspace of $\mathbb{R}^3$. Then

2.37 *dimension of a subspace*

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

$\dim(U) \leqslant 3$. Then $\dim(U) = 0, 1, 2$, or 3.

Case-I) If $\dim(U) = 0$.

We know that $\dim(\{0\}) = 0$. Here $\{0\}$ is the trivial subspace. Then $U = \{0\}$.

Case-II) If $\dim(U) = 1$.

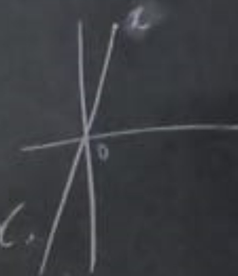
Then U has only one basis.

Case-II If $\dim(U)=1$,

then there exist only one basis vector in U (non zero)

Let say that $0 \neq x \in U$.

Then, $U=\{kx \mid k \in \mathbb{R}\}$



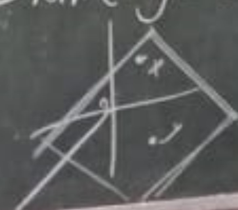
Thus, U is the line go through orign & x.

Case-III If $\dim(U)=2$.

Then there exist two non zero vectors x, y in basis of U.

Then $U=\{k_1 x + k_2 y \mid k_1, k_2 \in \mathbb{R}\}$

Thus, U is the plane go thrgh the x, y and the origin



$\boxed{\text{Case-IV}}$ If $\dim(U)=3$,

We know that U is subspace of $\mathbb{R}^3$ and

$$\dim(\mathbb{R}^3)=3=\dim(U)$$

Then $U=\mathbb{R}^3$

Therefore subspaces of $\mathbb{R}^3$ are

- $\{0\}$ (trivial subspace)
- All the line go throgh the orign
- All the planes go throgh the orign
- $\mathbb{R}^3$ (whole space)

3 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Let $U=\{p\in \mathcal{P}_4(\mathbb{F}) \mid p(6)=0\}$

$p_1=(x-6)$

$p_2=(x^2-6x)$

$p_3=(x^3-6x^2)$

$p_4=(x^4-6x^3)$

Claim 1. p_1, p_2, p_3, p_4 are linearly independent.

Suppose that there exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$0=a_1(x-6)+a_2(x^2-6x)+a_3(x^3-6x^2)+a_4(x^4-6x^3)$$
$$=(a_1-6a_2)x+(a_2-6a_3)x^2+(a_3-6a_4)x^3-6a_1$$

This implies, $-6a_1=0$

$$\boxed{a_1=0}$$

$$(a_1-6a_2)=0$$

$$\boxed{a_2=0}$$

Similary, $a_3=a_4=0$.

claim2: $p_1,\dots,p_4$ spans the V.

Let $p\in U$. Then $p(6)=0$. By Divison rule

$$p(x)=(x-6)(ax^3+bx^2+cx+d) \text{ for some } a,b,c,d\in \mathbb{F}.$$

$$= ax^4+bx^3+cx^2+dx$$
$$-6x^3-6bx^2-6cx-6d$$

$$=a(x^4-6x^3)+b(x^3-6x^2)+c(x^2-6x)$$
$$+d(x-6)$$

Therfore p_1,p_2,p_3,p_4 is a basis for U.

Let $U = \{ p \in \mathcal{P}_4(\mathbb{F}) \mid p(6) = 0 \}$

b)

$p_1 = (x-6)$

$p_2 = (x^2 - 6x)$

$p_3 = (x^3 - 6x^2)$

$p_4 = (x^4 - 6x^3)$

$p_5 = 1$

Claim 3: $p_1, \dots, p_5$ is a linearly indepent

Suppose that $\exists\, a_1, \dots, a_5 \in \mathbb{F}$ s.t.

$$a_1(x-6) + a_2(x^2 - 6x) + a_3(x^3 - 6x^2) + a_4(x^4 - 6x^3) + a_5 = 0$$

$$(a_5 - 6a_1) + (a_1 - 6a_2)x + (a_2 - 6a_3)x^2 + (a_3 - 6a_4)x^3 + 6a_4 = 0$$

Then $6a_4 = 0 \Longrightarrow a_4 = 0$

$(a_3 - 6a_4) = 0 \Longrightarrow a_3 = 0$

Similary $a_2 = a_1 = a_5 = 0$.

Thus $p_1, \dots, p_5$ are linearly independent

Claim 4: $p_1, \dots, p_5$ spans $\mathcal{P}_4(\mathbb{F})$

Let $p \in P_4(\mathbb{F})$ Then

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$= (ax^4 - 6ax^3) + (6ax^3 + bx^3 - 6(6a+b)x^2) + (6(6a+b)x^2$$

$$+ cx^2 - 6(36a+6b+c)x) + 6(36a+6b+c)x$$

$$+ 6(216a+36b+6c) - 6(216a+36b+6c) + d)$$

$$= a(x^4 - 6x^3) + (6a+b)(x^3 - 6x^2) + (36a+6b+c)(x^2 - 6x)$$

$$+ (216a+36b+6c$$

Note that, a, $(6a+b)$, $(36a+6b+c)$,

$(216a+36b+6c+d)$,

$(1296+216b+36c+6d+e) \in \mathbb{F}$.

Then P_1, P_2, P_3, P_4, P_5 spans $\mathcal{P}_4$

Therefore, $P_1 P_2, \ldots, P_5$ is a basis for $\mathcal{P}_4$

c) $W = \text{span}(1) = \{k \mid k \in \mathbb{F}\}$

Then $U + W = \mathcal{P}_4(\mathbb{F})$ (by part b))

Furthur, $U \cap W = \{0\}$.

Therfore $U \oplus W = \mathcal{P}_4(\mathbb{F})$